Active Fault Tolerant Control of Nonlinear Systems: the Cart–Pole example

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Abstract: This paper describes an active fault tolerant control scheme that is developed for nonlinear systems. The methodology is based on a fault detection and diagnosis procedure relying on adaptive filters designed via the nonlinear geometric approach. The controller reconfiguration exploits directly the on–line estimate of the fault signal. The classical model of an inverted pendulum on a cart is considered as an application example, in order to highlight the feasibility and the efficiency of the proposed approach.

Keywords: Fault detection and isolation, Nonlinear filter, Geometric approach, Fault–tolerant control, Cart–pole nonlinear model.

1. INTRODUCTION

Feedback control systems for mechatronics engineering applications strongly rely on actuators, sensors and data acquisition/ interface components to ensure a proper interaction between the physical controlled system and control devices. Faulty conditions of those system components lead to a drastic reduction or loss of stability and performance properties, which may even cause damages to the physical system. Therefore, there is a growing demand for reliability, safety and fault tolerance in control systems for mechatronics. It is necessary to design control systems which are capable of tolerating potential faults in order to improve the reliability and availability, while providing a desirable performance. A closed–loop control system which can tolerate component malfunctions, while maintaining desirable performance and stability properties is said to be a fault–tolerant control system. Over the last three decades, the growing demand for safety, reliability, and maintainability in technical systems has drawn significant research in Fault Detection and Diagnosis (FDD). Such efforts have led to the development of many FDD techniques, see for example the most recent survey works (Blanke et al. (2006); Isermann (2005); Witzczak (2007); Zhang and Jiang (2008)). In general, fault tolerant control methods are classified into two types, i.e. Passive Fault Tolerant Control Scheme (PFTCS) and Active Fault Tolerant Control Scheme (AFTCS) (Blanke et al. (2006); Zhang and Jiang (2008)).

This paper is focused on the development of a novel AFTCS, that integrates a reliable and robust fault diagnosis scheme with the design of a controller reconfiguration system. In particular, the methodology is based on a Fault Detection and Diagnosis (FDD) procedure relying on adaptive filters designed via the nonlinear geometric approach. The controller reconfiguration exploits a second control loop, depending on the on–line estimate of the fault signal. One of the advantages of this strategy is that, for example, a structure of logic–based switching controller is not required. The novelty of the proposed AFTCS lies in the feedback of the estimated fault signal, which is obtained by the adaptive filters designed via the nonlinear geometric approach. The achieved simulation results show how the closed loop of reconstructed fault signal not only enhances the feedback itself, but also improves the final performances of the overall system. Compared with different fault tolerant approaches, see e.g. (Marcos et al. (2005)), the suggested AFTCS strategy can maintain performance with significant actuator faults, since these signals are reconstructed by the FDD logic with good accuracy. Concerning the FDD procedure, the paper describes a nonlinear scheme, which provides the fault detection, the isolation and the fault size estimation. The FDD nonlinear method is based on the NonLinear Geometric Approach (NLGA) developed by De Persis and Isidori (De Persis and Isidori (2001)). By means of this framework, a disturbance de–coupled adaptive nonlinear filter providing the fault identification is developed. It is worth observing that the original NLGA FDD scheme based on residual signals cannot provide, in general, fault size estimation. Both the NLGA Adaptive Filters (NLGA–AF) and the AFTCS strategy are applied to the well–known model of an inverted pendulum on a cart (also called cart–pole system), an underactuated mechanical structure that is commonly used as a benchmark system for control design and mechatronics prototyping. A simulation model for the complete AFTCS loop has been implemented in the Matlab/Simulink® environments and tested in the

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presence of actuator faults, disturbances, measurement noise and modelling errors. The achieved results in faulty conditions show the asymptotic fault accommodation and the control objective recovery.

The work is organised as follows. Section 2 provides the description of the cart–pole nonlinear model. Section 3 describes the implementation of the FDD scheme and the structure of the AFTCS strategy. Section 4 gives a summary of the achieved results. Concluding remarks are summarised in Section 5.

2. THE CART–POLE NONLINEAR MODEL

The dynamic model of a pendulum (or pole) on a cart shown in Figure 1 is a classical benchmark in Systems and Control Theory.

Fig. 1. Scheme of the inverted pendulum on a cart.

The interest in this mechanical system is motivated by the similarity between its dynamic properties and those of several real–world engineering applications like, for example, aerospace vehicles during vertical take–off, cranes, and many others. Assuming that the cart has mass $M$, the pendulum mass $m$ is concentrated at the tip of a pole of length $L$, and that there are no friction effects, the dynamic model obtained using Hamilton’s principle is the following:

$$
\begin{align*}
(M + m)\ddot{x} + mL\dot{\theta}\cos{\theta} - mL\dot{\theta}^2\sin{\theta} &= F \\
m\ddot{x}\cos{\theta} + mL\dot{\theta} - mg\sin{\theta} &= \tau
\end{align*}
$$

(1)

in which $g$ is the gravity constant, whilst $F$ and $\tau$ are the linear force acting on the cart and the torque acting directly at the base of the pole, respectively. If the state variables are:

$$X = [x \ x \ \dot{x} \ \dot{\theta}]^T$$

(2)

and considering $u = F$ as the control input, and $d = \tau$ as a disturbance, the model can be rewritten in its state–space input affine form as follows:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{MLx_2^2\sin{x}_3 - mg\sin{x}_3\cos{x}_3 + u - \frac{d\cos{x}_3}{L}}{M + m\sin^2{x}_3} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{(M + m)g\sin{x}_3 - MLx_2^2\sin{x}_3\cos{x}_3}{(M + m\sin^2{x}_3)L} + \frac{u\cos{x}_3 - \frac{d}{L}}{(M + m\sin^2{x}_3)L}
\end{align*}
$$

(3)

As it can be seen, the dynamic model of the cart–pole system fulfils the structural requirements described in (De Persis and Isidori (2001)), which considers the following class of nonlinear systems:

$$\dot{X} = N(X) + G(X)u + P(X)d$$

(4)

where $N(X), G(X)$ and $P(X)$ are smooth vector fields. In the following section, the proposed solutions to FDD and AFTCS problems, based on the nonlinear geometric approach introduced by De Persis and Isidori, will be developed.

3. FDD DESIGN AND AFTCS SCHEME

This section describes the implementation of the FDD scheme and the structure of the AFTCS strategy. Regarding the presented FDD scheme, it belongs to the NLGA framework, where a coordinate transformation, highlighting a sub–system affected by the fault and decoupled by the disturbances, is the starting point to design a set of adaptive filters. They are able to both detect additive fault acting on a single actuator and estimate the magnitude of the fault. It is worth observing that, by means of this NLGA approach, the fault estimate is decoupled from disturbance $d$.

The proposed approach can be properly applied to the approximate nonlinear model of the system in the form:

$$
\begin{align*}
\dot{x} &= n(x) + g(x)c + \ell(x)f + p_d(x)d \\
y &= h(x)
\end{align*}
$$

(5)

where the state vector $x \in \mathcal{X}$ (an open subset of $\mathbb{R}^\ell$), $c(t) \in \mathbb{R}^c$ is the control input vector, $f(t) \in \mathbb{R}^f$ is the fault, $d(t) \in \mathbb{R}^d$ the disturbance vector (embedding also the faults which have to be de–coupled, in order to perform the fault isolation) and $y \in \mathbb{R}^m$ the output vector, whilst $n(x), \ell(x)$, the columns of $g(x)$, and $p_d(x)$ are smooth vector fields, with $h(x)$ is a smooth map.

The model (3) including an additive fault $f$ can be rewritten in the form:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{MLx_2^2\sin{x}_3 - mg\sin{x}_3\cos{x}_3 + (u + f)}{M + m\sin^2{x}_3} - \frac{d\cos{x}_3}{L} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{(M + m)g\sin{x}_3 - MLx_2^2\sin{x}_3\cos{x}_3}{(M + m\sin^2{x}_3)L} + \frac{(u + f)\cos{x}_3 + \frac{d}{L}}{(M + m\sin^2{x}_3)L}
\end{align*}
$$

(6)

where, with reference to the input–affine model (5), $x = [x_1 \ x_2 \ x_3 \ x_4]^T$, $c = u$, and with:

$$
n(x) = \begin{bmatrix}
\begin{bmatrix}
x_2 \\
MLx_2^2\sin{x}_3 - mg\sin{x}_3\cos{x}_3
\end{bmatrix} + \\
\begin{bmatrix}
(M + m)g\sin{x}_3 - MLx_2^2\sin{x}_3\cos{x}_3 \\
L(M + m\sin^2{x}_3)
\end{bmatrix}
\end{bmatrix}
$$

(7)
\[
g(x) \equiv \ell(x) = \begin{bmatrix}
0 \\
1 \\
M + m\sin^2 x_3 \\
0 \\
\cos x_3 \\
-\frac{L(M + m\sin^2 x_3)}{L}
\end{bmatrix}
\]

(8)

Moreover, \( p_d(x) \) is defined as:

\[
p_d(x) = \begin{bmatrix}
0 \\
\cos x_3 \\
-\frac{L(M + m\sin^2 x_3)}{L^2(M + m\sin^2 x_3)} \\
0 \\
1 \\
\end{bmatrix}
\]

(9)

The design of the strategy for the diagnosis of the fault \( f \) with disturbance de-coupling, by means of the considered NLGA, is organised as follows:

- computation of \( \Sigma_P^* \), i.e. the minimal conditioned invariant distribution containing \( P \) (where \( P \) is the distribution spanned by the columns of \( p_d(x) \));
- computation of \( \Omega^* \), i.e. the maximal observability codistribution contained in \( (\Sigma_P^*)^\perp \);
- if \( \ell(x) \notin (\Omega^*)^\perp \), fault detectability condition, the fault is detectable and a suitable change of coordinate can be determined.

\( \Sigma_P^* \) can be computed by means of the following recursive algorithm:

\[
\begin{align*}
S_0 &= \tilde{P} \\
S_{k+1} &= \tilde{S} + \sum_{i=0}^{m} \left[ g_i, \tilde{S}_k \cap \ker \{dh\} \right]
\end{align*}
\]

(10)

where \( m \) is the number of inputs, \( \tilde{S} \) represents the involutive closure of \( S, [g_i, \Delta] \) is the distribution spanned by all vector fields \([g, \tau]\), with \( \tau \in \Delta \), and \([g, \tau]\) the Lie bracket of \( g, \tau \). It can be shown that if there exists a \( k \geq 0 \) such that \( S_{k+1} = S_k \), the algorithm (10) stops and \( \Sigma_P^* = S_k \) (De Persis and Isidori (2001)).

Once \( \Sigma_P^* \) has been determined, \( \Omega^* \) can be obtained by exploiting the following algorithm:

\[
\begin{align*}
Q_0 &= (\Sigma_P^*)^\perp \cap \ker \{dh\} \\
Q_{k+1} &= (\Sigma_P^*)^\perp \cap \sum_{i=0}^{m} [L_q, Q_k + \ker \{dh\}]
\end{align*}
\]

(11)

where \( L_q \Gamma \) denotes the codistribution spanned by all covector fields \( L_q \omega \), with \( \omega \in \Gamma \), and \( L_q \omega \) the derivative of \( \omega \) along \( g \).

If there exists an integer \( k^* \) such that \( Q_{k^*} = Q_{k^*+1} \), \( Q_{k^*} \) is indicated as o.c.a. \( (\Sigma_P^*)^\perp \), where o.c.a. stands for observability codistribution algorithm. It can be shown that \( Q_{k^*} = o.c.a. (\Sigma_P^*)^\perp \) represents the maximal observability codistribution contained in \( P^\perp \), i.e. \( \Omega^* \) (De Persis and Isidori (2001)). Therefore, with reference to the model (5), when \( \ell(x) \notin (\Omega^*)^\perp \), the disturbance \( d \) can be de-coupled and the fault \( f \) is detectable.

As mentioned above, the considered NLGA to the fault diagnosis problem, described in (De Persis and Isidori (2001)), is based on a coordinate change in the state space and in the output space, \( \Phi(x) \) and \( \Psi(y) \), respectively. They consist in a surjection \( \Psi \) and a function \( \Phi_1 \) such that \( \Omega^* \cap \ker \{dh\} = \ker \{d(\Psi_1 \circ h)\} \) and \( \Omega^* = \ker \{d\Phi_1\} \), where:

\[
\begin{align*}
\Phi(x) &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix} \\
\Psi(y) &= \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \Psi_1(y) \\ H_2 y \end{bmatrix}
\end{align*}
\]

(12)

are (local) diffeomorphisms, whilst \( H_2 \) is a selection matrix, i.e. its rows are a subset of the rows of the identity matrix. By using the new (local) state and output coordinates \((\bar{x}, \bar{y})\), the system (5) is transformed as follows:

\[
\begin{align*}
\dot{\bar{x}}_1 &= n_1(\bar{x}_1, \bar{x}_2) + g_1(\bar{x}_1, \bar{x}_2) c + \ell_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) f \\
\dot{\bar{x}}_2 &= n_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) + g_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) c + + \ell_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) f + p_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) d \\
\dot{\bar{x}}_3 &= n_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) + g_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) c + + \ell_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) f + p_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) d \\
\ddot{\bar{y}}_1 &= h(\bar{x}_1) \\
\ddot{\bar{y}}_2 &= \bar{x}_2
\end{align*}
\]

(13)

with \( \ell_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) \) not identically zero. As described in (De Persis and Isidori (2001)), in this way the observable subsystem (13) which, if it exists, is affected by the fault and not affected by disturbances, and the other faults to be de-coupled, is obtained. This transformation can be applied to the system (5) if and only if a the fault detectability condition is satisfied. The system (5) in the new reference frame can be decomposed into 3 subsystems (13) where the first one (the so-called \( \bar{x}_1 \)-subsystem) is always de-coupled from the disturbance vector and affected by the fault as follows:

\[
\begin{align*}
\dot{\bar{x}}_1 &= n_1(\bar{x}_1, \bar{y}_2) + g_1(\bar{x}_1, \bar{y}_2) c + \ell_1(\bar{x}_1, \bar{y}_2, \bar{x}_3) f \\
\ddot{\bar{y}}_1 &= h(\bar{x}_1)
\end{align*}
\]

(14)

where, as the state \( \bar{x}_2 \) in (13) is assumed to be measured, the variable \( \bar{x}_2 \) in (14) is considered as independent input and denoted with \( \bar{y}_2 \).

In the case of (6), with reference to (5), and recalling (9) and (8), the following is obtained:

\[
\begin{align*}
S_0 &= \tilde{P} = cl(p_d(x)) = 0 \\
S_{k+1} &= \tilde{S} + \sum_{i=0}^{m} [L_q, Q_k + \ker \{dh\}]
\end{align*}
\]

(15)

By recalling that \( \ker \{dh\} = 0 \), it follows that \( \Sigma_P^* = \tilde{P} \) as \( S_0 \cap \ker \{dh\} = 0 \). Thus, the algorithm (10) stops with \( S_1 = S_0 = \Sigma_P^* \).

On the other hand, in order to solve (11), it is necessary to compute the expression \( (\Sigma_P^*)^\perp = (\tilde{P})^\perp \). However, it is worth noting that, for the case under investigation, the determination of the codistribution \( (\Sigma_P^*)^\perp = (\tilde{P})^\perp \) is enhanced due to the sparse structure of (15). Moreover, by means of (11), the computation of \( (\Sigma_P^*)^\perp = (\tilde{P})^\perp \) leads to a codistribution \( \Omega^* = o.c.a. (\Sigma_P^*)^\perp \) spanned by exact differentials. Finally, any codistribution \( \Omega \) which is
a conditioned invariant contained in $P^\perp$ spanned by exact differentials, with $\Omega = \text{o.c.a.}(\Omega)$ and $\ell(x) \notin (\Omega)^\perp$ can be used to define the coordinate change (12). Therefore, the computation of the maximal observability codistribution is not required.

By observing that:

$$
(\bar{P})^\perp = \begin{pmatrix}
0 & \cos x_3 \\
L(M + m \sin^2 x_3) & 0 \\
L_2(M + m \sin^2 x_3) & 1 \\
1 & 0 \\
0 & 0 \\
0 & 1 - Lx_4 \sin x_3 \\
0 & L \cos x_3
\end{pmatrix}^\perp
$$

(16)

and noting that span $\{dh\} = I_4$, from (11) it follows that $\Omega^\perp = (\Omega^\perp)^\perp = (P)^\perp$, and $\Omega^\perp = \bar{P} \circ P$. The fault in (6) is detectable if $\ell(x) \notin (\Omega^\perp)^\perp = \bar{P} \circ \bar{P} = \bar{P}$. This condition is fulfilled due to the expression of $\ell(x)$ in (8).

As dim $\{\Omega^\perp\} = 3$, and dim $\{\Omega^\perp \cap \text{span} \{dh\}\} = 3$, it follows that $\Phi_1(x) : \mathbb{R}^4 \to \mathbb{R}^4$. Moreover, as $\Omega^\perp \cap \text{span} \{dh\} = \text{span} \{d(\Phi_1 \circ h)\}$, $H_2 y : \mathbb{R}^4 \to \mathbb{R}^4$. Thus, as $h(x) = I_x x$, the surjection $\Psi(y(x))$ is given by:

$$
\Psi(y(x)) = \begin{pmatrix}
\Psi_1(x) \\
\Psi_2(x) \\
\Psi_3(x)
\end{pmatrix}
$$

(17)

where $H_2 = [0 \ 0 \ 0 \ 1]$. Note that, since $dh = I_4$, the diffeomorphism $\Phi_1(x)$ such that $\Omega^\perp = \text{span} \{d(\Phi_1)\}$ is given by:

$$
\Phi_1(x) = \Psi_1(y(x)) = \Psi_1(x)
$$

(18)

Hence, the $\bar{x}_1$-subsystem state variable is:

$$
\bar{x}_1 = \begin{pmatrix}
\bar{x}_{11} \\
\bar{x}_{12} \\
\bar{x}_{13}
\end{pmatrix} = \begin{pmatrix}
x_2 + Lx_4 \cos x_3 \\
x_1 \\
x_3
\end{pmatrix}
$$

(19)

It is worth observing that only $\bar{x}_{11}$ is affected by the faults, and that the differentials of $x_2 + Lx_4 \cos x_3$ span an observability codistribution $\Omega$ contained in $P^\perp$ with $\Omega = \text{o.c.a.}(\Omega)$. Hence, as previously remarked, in order to estimate the fault, it is possible to use the scalar subsystem defined by the coordinate $\bar{x}_{11} = x_2 + Lx_4 \cos x_3$, whose dynamics are defined by:

$$
\dot{\bar{x}}_{11} = \frac{d(x_2 + Lx_4 \cos x_3)}{dt}
$$

(20)

from which, by assuming that the whole state is measured, the NLGA–AF can be computed.

With reference to (14), the NLGA–AF can be designed if the condition in (De Persis and Isidori (2001)) and the following new constraints are satisfied:

- the $\bar{x}_1$-subsystem is independent from the $\bar{x}_3$ state components;
- the fault is a step function of the time, hence the parameter $f$ is a constant to be estimated;

- there exists a proper scalar component $\bar{x}_{1s}$ of the state vector $\bar{x}_1$ such that the corresponding scalar component of the output vector is $\bar{y}_{1s} = \bar{x}_{1s}$ and the following relation holds (Bonf`e et al. (2007)):

$$
\hat{y}_{1s}(t) = M_1(t) \cdot f + M_2(t)
$$

(21)

where $M_1(t) \neq 0, \forall t \geq 0$. Moreover $M_1(t)$ and $M_2(t)$ can be computed for each time instant, since they are functions just of input and output measurements. The relation (21) describes the general form of the system under diagnosis. Under these conditions, the design of the adaptive filter is achieved, with reference to the system model (21), in order to provide a fault estimation $\hat{f}(t)$, which asymptotically converges to the magnitude of the fault $f$. The proposed adaptive filter is based on the least–squares algorithm with forgetting factor (Ioannou and Sun (1996)), and it is described by the following adaptation law:

$$
\begin{cases}
\dot{\hat{y}}_{1s} = \hat{M}_1 \hat{f} + \hat{M}_2 + \lambda \hat{y}_{1s} \\
\epsilon = \frac{1}{N^2} (y_{1s} - \hat{y}_{1s})
\end{cases}
$$

(23)

where all the involved variables of the adaptive filter are scalar. In particular, $\lambda > 0$ is a parameter related to the bandwidth of the filter, $\beta \geq 0$ is the forgetting factor and $N^2 = 1 + \hat{M}_2^2$ is the normalisation factor of the least–squares algorithm. Moreover, the proposed adaptive filter adopts the signals $\hat{M}_1$, $\hat{M}_2$, $\hat{y}_{1s}$ which are obtained by means of a low–pass filtering of the signals $M_1$, $M_2$, $y_{1s}$ as follows:

$$
\begin{cases}
\hat{M}_1 = -\lambda \hat{M}_1 + M_1 \\
\hat{M}_2 = -\lambda \hat{M}_2 + M_2
\end{cases}
\quad
\begin{cases}
\hat{y}_{1s} = -\lambda \hat{y}_{1s} + y_{1s} \\
\hat{y}_{1s} = \hat{y}_{1s} + \hat{y}_{1s}
\end{cases}
$$

(24)

Thus, the considered adaptive filter is described by the systems (22), (23), and (24). It can be proved that the asymptotic relation between the normalised output estimation error $\epsilon(t)$ and the fault estimation error $f - \hat{f}(t)$ is the following:

$$
\lim_{t \to \infty} \epsilon(t) = \lim_{t \to \infty} \frac{\hat{M}_1(t)}{N^2(t)} (f - \hat{f}(t))
$$

(25)

Moreover, it can be proved that the adaptive filter described by the relations (22), (23), and (24) provides an estimation $\hat{f}(t)$ that asymptotically converges to the magnitude of the step fault $f$. The proofs above have been omitted due to lack of space.

In order to design the NLGA–AF scheme, it is possible to design 4 NLGA adaptive filter in the form of (22), (23), and (24), allowing to estimate the magnitude of a step fault acting on the linear force actuator of the inverted pendulum, as shown in (6). In order to de–couple the effect of the disturbance $d$ from the fault estimator, it is
necessary to select from the \( \ddot{x}_1 \)–subsystem the following state component:

\[
\ddot{x}_{1s} = \dddot{x}_1 = x_2 + Lx_4 \cos x_3
\]  

(26)

Hence, it is possible to specify the specific expression of the fault dynamics (21). The design of the NLGA–AF for \( f \) is based on:

\[
\begin{aligned}
\dot{y}_{1s} &= M_1 f + M_2 \frac{1 - \cos^2 x_3}{M + m \sin^2 x_3} \\
M_1 &= \frac{1 - \cos^2 x_3}{M + m \sin^2 x_3} \\
M_2 &= \frac{MLx_4^2 \sin x_3 - mg \sin x_3 \cos x_3}{M + m \sin^2 x_3} + \\
&\quad + \frac{(M + m) \sin x_3 \cos^2 x_3}{M + m \sin^2 x_3} \\
&\quad - \frac{mx_3^2 \sin x_3 \cos x_3}{M + m \sin^2 x_3} + \\
&\quad - x_3^2 \sin x_3 + \frac{1 - \cos^2 x_3}{M + m \sin^2 x_3} u
\end{aligned}
\]  

(27)

To compute the simulation results described in next section, the AFTCS scheme has been completed by means of an optimal state feedback control law \( u = -K \dot{x} \), designed on the basis of the linear approximation of model (3) in a neighbourhood of \( \dot{x}_n = [x_{1r} \ 0 \ 0 \ 0]^T \), in which \( x_{1r} \) can be any value. In fact, the linear approximation is independent from \( x_1 \), so that the input vector of optimal controller can be calculated as \( \dot{x} = [x_1 - x_{1r}] \ x_2 \ x_3 \ x_4 \) and the cart–pole system will be stabilised at the upright position by using the logic scheme represented in Figure 2.

\[
\begin{array}{c}
\text{Optimal control} \\
\downarrow
\end{array}
\begin{array}{c}
x_{1d}
\end{array}
\begin{array}{c}
\text{Cart-pole dynamics}
\end{array}
\begin{array}{c}
u
\end{array}
\begin{array}{c}
\text{FDD module}
\end{array}
\begin{array}{c}
\dot{f}
\end{array}
\begin{array}{c}
\hat{f}
\end{array}
\]

Fig. 2. Logic diagram of the integrated AFTCS strategy.

With reference to Figure 2, the following nomenclature and symbols have been used:

- \( x_{1r} \), the desired value of the linear position;
- \( u \), actuated input;
- \( u_c \), control input;
- \( u_i \), output signal from the optimal controller;
- \( y \), measured output;
- \( f \), actuator fault;
- \( \hat{f} \), estimated actuator fault.

Therefore, the logic scheme depicted in Figure 2 shows that the AFTCS strategy is implemented by integrating the FDD module with the existing control system. From the controlled input and output signals, the FDD module provides the correct estimation \( \hat{f} \) of the \( f \) actuator fault, which is injected to the control loop, in order to compensate the effect of the actuator fault. After this correction, the optimal controller provides the exact tracking of the reference signal \( x_{1r} \). The simulation results will show that the feedback of the estimated fault \( \hat{f} \) improves the identification of the fault signal \( f \), itself, by reducing also the estimation error and possible bias due to the model–system mismatch. The formal proof of the stability of the overall AFTCS will be investigated in further works. However, simulation results highlight that the model state variables remain bounded in a set, which assures control performance, even in the presence of large fault sizes. Moreover, the assumed fault conditions do not modify the system structure, thus guaranteeing the global stability. Finally, Section 4 will show the simulation results that have been achieved by implementing the presented integrated FDD and AFTCS strategy. However, the asymptotic fault accommodation, and the control objective recovery, that in this paper are assessed in simulation, will require further studies and investigations.

4. SIMULATION RESULTS

To show the diagnostic characteristics brought by the application of the proposed AFTCS and FDD schemes to the inverted pendulum on a cart, the nonlinear dynamic model of the mechanical system has been implemented in Matlab/Simulink®. The following values of the system parameters have been assumed: \( M = 1 \) kg; \( m = 0.1 \) kg; \( L = 0.3 \) m; \( g = 9.81 \) m/s².

The optimal controller has been designed using the LQR approach in order to minimize the cost function:

\[
J = \int_0^{+\infty} (X_\ell^T Q X_\ell + uRU)dt
\]  

(28)

with \( Q = 10 I_4 \) and \( R = 1 \).

In order to show the capabilities of the proposed AFTCS strategy, the system has been tested setting as a reference \( x_{1r} \) a square–wave of 0.2 m, amplitude and 50 seconds period. A random disturbance \( d \) modelled as a zero–mean band–limited noise has been applied. It is worth noting that the filter is structurally de–coupled from this disturbance torque, while the measurement noise and the modelling errors may affect the fault estimation. The following results refer to the simulation of a fault \( f \) modelled as a step signal with a size of 0.1 N., commencing at \( t = 66s \). Figure 3 shows the estimate of the actuator fault \( f \) (solid line), when compared with the simulated actuator fault (dashed line). The fault estimate has been achieved by using the logic scheme represented in Figure 2, and from the FDD module described in Section 3. As it can be seen, after a suitable choice of the parameters of the fault (22), (23), and (24), the FDD module provides a quite accurate estimate of the fault size, with minimal detection delay. Residual errors are due to both the measurement
noise and the mismatch between the parameters in the plant and those in the NLGA adaptive filters.

![Fault Estimation](image)

**Fig. 3.** Real-time estimate \( \hat{f} \) of the actuator fault \( f \).

Figure 4 shows the cart position \( x_1 \) compared with its desired value \( x_{1r} \). When the fault is not acting on the system, the position error is quite small and is affected mainly by the disturbance torque \( d \). The fault commences at \( t = 66 \text{s.} \), but the fault estimate feedback is applied after \( t = 110 \text{s.} \).

![Cart position \( x_1 \)](image)

**Fig. 4.** Linear position \( x_1 \) of the cart in the fault–free and faulty cases, with and without AFTCS.

As highlighted in Figure 4, during the time interval 66s. < \( t < 110 \text{s.} \), the steady–state error cannot be eliminated by the optimal controller without AFTCS. On the other hand, when the proposed AFTCS scheme is switched on, the steady–state error due to the fault is almost zero. The achieved simulation results summarised in Figures 3 and 4 show the effectiveness of the presented integrated FDD and AFTCS strategy, which is able to improve the control objective recovery, and the reference tracking in the presence of actuator fault. However, the asymptotic fault accommodation, the transient and the asymptotic stability of the controlled system, that in this paper are assessed in simulation, require further theoretical studies and investigations. Finally, it is worth noting that the suggested NLGA–AF provides not only FDI but also a fault estimate. For this reason, it could be compared e.g. with the fault identification scheme proposed in (Kaboré and Wang (2001); Kaboré et al. (2000)). However, the proposed NLGA–AF is less sensitive to measurement noise, which allows to obtain a smaller minimal detectable fault. On the other hand, the fault estimation technique of (Kaboré and Wang (2001)) can provide a faster response and, therefore, a slower detection time.

5. CONCLUSION

This paper described the development of a novel active fault tolerant control scheme, which integrates a robust fault diagnosis scheme with the design of a controller reconfiguration system. The methodology was based on a fault detection and diagnosis procedure relying on adaptive filters designed via the nonlinear geometric approach. The novelty of the proposed fault tolerant scheme consisted of the use of the fault signal estimated by these adaptive filters and exploited in the closed loop scheme for improving the performances of the overall system. The fault tolerant strategy has been applied to a classical control design benchmark, namely the inverted pendulum on a cart, which has been simulated in presence of actuator faults, disturbing forces, measurement noise and modelling errors. Further investigations will regard the proof of the stability of the complete fault tolerant scheme.

REFERENCES


