Feedback Linearization Control of a Class of Nonlinear Systems
Based on Fuzzy Inference Systems

M. Bahita*. K. Belarbi **

*Faculty of Hydrocarbons and Chemistry, University of Boumerdes, Boumerdes 35000 Algeria (Tel: 213-31-819010; e-mail: mbahita@yahoo.fr).

**Constantine University, Faculty of Engineering, 25000 Constantine (e-mail: kbelarbi@yahoo.fr)

Abstract: In this work, we consider the application of an adaptive fuzzy control for a class of single input single output nonlinear systems. The method uses a fuzzy inference system of Takagi Sugeno (TS) type to approximate the feedback linearization law and a fuzzy inference system of Mamdani type to estimate the control signal error which is not available. The parameters of the (TS) controller are adapted and changed using the gradient descent law based on the estimated control error. The rule base of the Mamdani fuzzy inference system is constructed from expert knowledge. The global stability of the resulting closed loop system is guaranteed using Lyapunov theory. The simulation results for an unstable nonlinear plant demonstrate the control effectiveness of the proposed algorithm.

Keywords: Adaptive control, Fuzzy inference system, Nonlinear systems, feedback linearization, Lyapunov theory.

1. INTRODUCTION
Adaptive control of linear systems and certain special classes of nonlinear systems has been well developed from the late 1970 to the 1990 [Feng, Gang 2006]. While adaptive control of general nonlinear systems still presents a challenge to control community. Nevertheless, mathematical models might not be available for many complex systems in practice, and the adaptive control problem of these systems is far from being satisfactorily resolved. Fuzzy logic systems [Wang, L. X 1993, Sanner, S. M. and Slotine, J. E. 1992] are widely used for this purpose. The key idea of these works is to use fuzzy systems to approximate unknown nonlinear functions in nonlinear systems and to represent the fuzzy systems in the form of linear regression with respect to unknown parameters and then to apply the well developed adaptive control techniques. Based on this, a great number of works on adaptive fuzzy control have been reported [Wang, L.-X. and Mendel, J. M. 1992, Castro, J. L. 1995, Hornick, K., Stinchombe, M. and White, H. 1989, Funahashi, F. L. 1989].

theory and the tracking error is used as an adaptation signal. Methods based on the minimisation of a cost function for obtaining weights update are generally considered to be not applicable or at least difficult to implement because they may yield sensitivity functions that are not computable [Ioannou P. A., and Sun, J. 1996], particularly if the cost function involves the control error since the ideal control law is not known. Nonetheless, a few authors have proposed fuzzy or neural network adaptive control based on the minimisation of this very objective. Indeed, in [Anderson, H. C., Lotfi, A., and Tsoi, A. C. 1997] a fuzzy controller is first designed off line for the plant, adaptation is then introduced on line. The control error appearing in the updating law is computed using the off line designed fuzzy controller. In [Chemori, A. 2001] a fuzzy estimator of the control error is used for on line adaptation of a heuristic CMAC controller. In both works, however, no stability results are provided. More recently [Labiod, S. and Guerra, T. M. 2007], a stable adaptive fuzzy control based on the control error is obtained as a function of the tracking error using a linear approximation of the nonaffine plant model. In this work, we build on the initial proposal [Chemori, A. 2001] to construct a fuzzy direct adaptive control of SISO nonlinear system without use of a compensatory (supervisory) control term. Based on some a priori knowledge of the plant, a Mamdani fuzzy logic estimator is constructed for estimating the control error. The parameter update are then computed using the estimated error and Lyapunov theory is used to derive guaranteed stability results for the closed loop.

This paper is organised as follows, the control problem is described in section 3, the stability of the closed loop is guaranteed using Lyapunov theory and the tracking error is used as an adaptation signal. More specifically, determine a feedback control $u = u_c(x, \theta)$ based on fuzzy logic systems and an adaptive low for adjusting the parameter vectors $\theta$ such that the closed-loop system must be globally stable and the tracking error, $e = y_a - y$ should be as small as possible. Define the error vector as

$$ e = (e_1, e_2, \ldots, e_n)^T \in \mathbb{R}^n $$

Step 1:
We chose $u$ to cancel the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form, and guarantee tracking convergence. If the functions $f(x)$ and $g(x)$ are known and the external disturbance $d$ does not exist, then from (2), the control law is

$$ u^* = \frac{1}{g'(x)} (v - f(x)) $$

Substituting (4) into (1), we can cancel the nonlinearities and obtain the simple input-state relation

$$ x(n) = v $$

Step 2:
We chose the artificial input $v$ (an equivalent input) as a simple linear pole-placement controller $v = y^{(n)\alpha} + K^T e$ that provides guarantee about the stability of the overall system, with the $k = (k_0, k_1, \ldots, k_{n-1})^T \in \mathbb{R}^n$ be such that the polynomial $h(x) = s^n + k_{n-1}s^{n-1} + \ldots + k_0 = 0$ has all its roots strictly in the left-half complex plane. Then the feedback linearization control law obtained from certainty equivalence approach is derived in [Slotine, J. E. and Weiping, L. 1991] as:

$$ u^* = \frac{1}{g'(x)} (y^{(n)\alpha} + K^T e - f(x)) $$

Applied to (2) results in

$$ e(n) + k_{n-1} e^{(n-1)} + \ldots + k_0 e = 0 $$

this implies that $\lim_{t \to \infty} e(t) \to 0$ (exponentially stable dynamics), which is the main objective of control. Since $f(x)$ and $g(x)$ are unknown and the external disturbance $d$ exists, the ideal control $u^*$ of (6) can not be implemented. In the next section, we propose a direct fuzzy adaptive control system to approximate this ideal control law.

3. THE DIRECT ADAPTIVE FUZZY CONTROLLER

The fuzzy adaptive controller proposed is composed of two fuzzy systems, a fuzzy controller based on a Takagi Sugeno (TS) inference system and a fuzzy estimator obtained through expertise and based on Mamdani inference system. This latter is suitable when trying to translate operator knowledge into fuzzy rules.

3.1 The TS fuzzy adaptive controller
A TS fuzzy inference system with linear consequences is composed of rules of the form:

\[ R^i : \text{if } x_i \text{ is } A_i^l \text{ and ... } x_n \text{ is } A_n^l \text{ then } u_i = a_i^0 + a_i^1 x_1 + ... + a_i^nx_n \]

where \( a_i^0, a_i^1, ..., a_i^n \) are fuzzy sets. If we take \( \theta^i = [a_i^0, a_i^1, ..., a_i^n] \) as the vector of adjustable parameters of the consequence of rule \( R^i \), the output of a TS fuzzy system can be put in the following form:

\[ u(x) = \sum_{i=1}^{n} \theta^i \tilde{z}(x) \]  

(8)

where \( \tilde{z}(x) = [\tilde{z}_1(x), ..., \tilde{z}_m(x)] \) is a vector of fuzzy basis functions. It has been proven that (8) can approximate over a compact set \( \Omega_z \) any smooth function up to a given degree of accuracy [Wang, L.-X. and Mendel, J. M. 1992, Castro, J. L. 1995].

Since both \( \tilde{z}_i(x) \) and \( \theta^i \) are positive, they can be lumped to \( \sum_{i=1}^{n} \theta^i \tilde{z}(x) \) and thus the learning rate and:

\[ \frac{\partial}{\partial \theta^i} \tilde{z}(x) = \tilde{z}(x) \]  

(9)

where \( \tilde{z}[\theta] = [\tilde{z}_1(\theta), ..., \tilde{z}_m(\theta)] \) contains all adjustable parameters and \( \tilde{w}(\theta) \) is a vector of fuzzy basis functions. It is a design constant.

3.2 The fuzzy estimator of the control error

In this paragraph, we develop the analysis for obtaining the fuzzy estimator of the control error. We first identify the situations where the control signal is “correct” in the sense that it is driving the output towards the reference. Without any particular knowledge on the system, it is clear that this occurs when the tracking error is maintained at zero or when it is decreasing. On the other hand, the control signal is clearly not correct when the output is drifting away from the reference. In this case we need to have some a priori knowledge about the controlled system. From an operator point of view, we can distinguish the following possibilities:

- There are systems where the control signal and output have the same direction. In these systems, increasing the control signal yields an increase in the output and the control signal yields a decrease in the output.
- There are systems that may behave in either way depending on the operation point. In these systems, decreasing the control signal yields an increase in the output and increasing the control signal yields a decrease in the output.
- There are systems that may behave in either way depending on the operation point.

The first two types of systems have a monotonic behaviour, in the sequel we consider only these systems.

Considering the first type of systems, two cases may arise when the control signal is not correct. In the first case, the output may be drifting away from the reference from above, in which case the magnitude of the applied control signal \( u(c) > u_{\text{Fopt}} \) is larger than \( u_{\text{Fopt}} \) and thus \( u_{\text{Fopt}} - u(c) < 0 \). In the second, it may be drifting away from below and the magnitude of the applied control signal is less than \( u_{\text{Fopt}} \) and thus \( u_{\text{Fopt}} - u(c) > 0 \).

Having characterised the control error, we are now ready to construct the rule base of the fuzzy estimator. The output of the FIS is the crisp value of the estimated control error. The crisp input variables are the current tracking error \( e(t) \) and the change of the tracking error \( de(t) = e(t) - e(t-1) \). Introducing the fuzzy variables: \( \text{ERROR}(e) \), \( \text{VARIATION OF ERROR}(d) \) and \( \text{CONTROL ERROR}(\hat{e}) \), each taking three fuzzy values: \( \text{ZERO} \) (or Z), \( \text{NEGATIVE} \) (or N) and \( \text{POSITIVE} \) (or P). We obtain the rule base, RB1, for the monotonic systems of the first type based on the following three cases:

Case1: (the control signal is correct when the tracking error is zero or decreasing)
Case2: (the output is drifting away from above)
Case3: (the output is drifting away from below)

All this in three cases induces the following rules:
Similarly, rule base RB2 for systems of the second type can be obtained in a similar manner, case1 remaining the same, while the consequences of case2 and case3 are interchanged. From the above analysis, it is clear that this procedure provides an estimated control error \( \hat{e}_n \) with the correct sign. In order to completely define the fuzzy estimator, we first choose the shape of all the membership functions and their distribution on the universe of discourse as shown in Fig. 2, where \( c_N, c_Z, c_P \), are the points where the membership functions reach their maximum. After fuzzification and \( \text{(prod, max)} \) inference strategy, the crisp estimated control error \( \hat{e}_n \) is computed through center of gravity defuzzification formula.

![Fig. 1 Distribution of membership functions on the universe of discourse](image)

Remark. From the computational and practical point of view, the number of fuzzy sets is chosen so that we can cover the universe of discourse (variation interval) of each variable.

4. STABILITY ANALYSIS

In order to analyze the stability of the fuzzy adaptive controller, we first derive the error equation. Inserting \( u_c(z, \theta) \) in the system equation (1) and based on equations (9) and (11), we replace it by:

\[
u_c(z, \theta) = u_{ref} - e_v - e - e_n\]  

(18)

Adding and subtracting also \( g(\chi)u_r \) to (1), using (6) and after some simple calculations, the error equation governing the closed loop system is:

\[\dot{e} = A_e \cdot e - B(d - g(\chi)(e_v + e_n))\]  

(19)

Where \( d \) is a bounded disturbance and

\[
A_e = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-k_0 & -k_1 & -k_2 & \ldots & -k_{n-2} & -k_{n-1}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

(20)

Define the Lyapunov function:

\[V = \frac{1}{2}e^T \cdot P \cdot e\]  

(21)

where \( P \) is a solution of the Lyapunov equation:

\[A_e^T \cdot P + PA_e = -Q\]  

with \( Q > 0 \).

Differentiate \( V \) with respect to time:

\[\dot{V} = \frac{1}{2}e^T \cdot P \dot{e} + \frac{1}{2}e^T \cdot P \dot{e}\]  

(23)

using (19) and (22), we obtain:

\[\dot{V} = -\frac{1}{2}e^T \cdot Q \cdot e - e^T \cdot P \cdot B \cdot (d - g(\chi)(e_v + e))\]  

(24)

or:

\[\dot{V} = -\frac{1}{2}e^T \cdot Q \cdot e - e^T \cdot P \cdot B d_i\]  

(25)

Lemma:

Consider the nonlinear plant (1) (2) with the control law given by (8) and updating law given by (17), then, the overall scheme guarantees that: if \( d_i \) in (25) is squared integrable, then is \[\int_0^\infty \norm{d_i(t)} dt < \infty\], then \[\lim_{t \to \infty} \norm{e(t)} = 0\]

Proof of Lemma:

From above we have \( d_i = d - g(\chi)(e_v + e) \) where \( e \) can be made arbitrarily small by using an appropriate number of fuzzy approximators [Wang, L.-X. and Mendel, J. M. 1992, Wang, L. X. 1993], \( g(\chi) \) and disturbance \( d \) are assumed to be bounded, the key element in \( d_i \) is the control error \( e_v \). From section 3, we know that if the step size is appropriately chosen, the control error \( e_v \) remains bounded. Hence the quantity \( d_i \) is bounded. Based on Babalat’s Lemma [Slotine, J. E. and Weiping, L.1991] then \( \lim_{t \to \infty} \norm{e(t)} = 0 \). Proof of this lemma is similar to part of a theorem proof of Wang’s work [Wang, L. X. (1993)].

5. SIMULATION RESULTS

In this section, we test the performance of the proposed fuzzy adaptive control law on the inverted pendulum system. The Dynamic equations of the inverted pendulum system are:

\[x_1 = x_2\]

\[\dot{x}_2 = f(\chi) + g(\chi)u(t) + d(t)\]  

(26)

\[y = x_1\]

Where

\[f(\chi) = \frac{g \sin(x_1) - \frac{ml^2 \sin(x_1) \sin(x_2)}{M+m}}{\frac{M+m}{M+m}}\]  

\[g(\chi) = \frac{\cos(x_1) \frac{M+m}{M+m}}{\frac{M+m}{M+m}}\]  

(27)

\[x_1 = \theta\] is the angular position of the pendulum, \( x_2 = \dot{\theta}\) is the angular velocity of the pendulum. We use \( g = 9.8 \text{m/s}^2 \), \( M = 1 \text{kg} \) is the mass of the cart, \( m = 0.1 \text{kg} \) is the mass of the cart.
pole and \( l = 0.5m \) is the half length of the pole. The control objective is to make the pole of the pendulum track a sine wave trajectory \( y_m = AM \sin(t) \) with different amplitudes, \( AM \), and we terminate by balancing the pole to the vertical position \((x_1, x_2) = (0,0)\), i.e., \( AM = 0 \).

Clearly, the derivatives of the reference \( y_m \) exist and are bounded. The external disturbance \( d \) is assumed to be a sine wave with amplitude \( \pm 0.1 \), period \( 2\pi \), and the parameters are chosen as \( \gamma = 1.4 \), \( v_0 = 0.005 \), step size \( dt = 0.01 \), and \( k = [k_0 \ k_1] = [35 \ 5]^T \) in order to have all roots of \( s^2 + k_1s + k_0 = 0 \) in the open left-half plane. The TS fuzzy controller has three inputs \( \varepsilon = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3] = [\theta \ \dot{\theta} \ (k^T \varepsilon + \theta^{(2)})] \) with \( \varepsilon = [\theta - \dot{\theta} \ \dot{\theta} - \ddot{\theta}] \). All three inputs are fuzzified with two fuzzy sets and similar membership functions given by:

\[
\mu_N(x) = \exp(-(x + 2)^2/2) , \quad \mu_P(x) = \exp(-(x - 2)^2/2) \quad (28)
\]

This gives eight rules of the form:

\[
R^i \text{ if } z_1 \text{ is } A^i_1 \text{ and } z_2 \text{ is } A^i_2 \text{ and } z_3 \text{ is } A^i_3 \text{ then }
\]

\[
u_i = a^i_1 z_1 + a^i_2 z_2 + a^i_3 z_3
\]

(29)

We have 24 parameters to tune. All parameters are initialised to zero. The fuzzy estimator is as described in section 3, with rule base RB1 since our system is of the first type described in section 3. The parameters defining the membership functions as shown in Fig. 1 are set to \( c_N = -2, c_E = 0, \ c_P = 2 \) for the error, the variation of error and for the estimated control error. The simulation results for two different amplitudes of the reference signal are shown in Fig 2 to 5, where the system output \( y(t) \) (pole angle) is in dashed while the reference signal \( y_m(t) \) is in continuous. Fig 2 shows the response curve of the pole angle from the initial position (-0.2, 0) and the corresponding desired values with amplitudes, \( AM = \pi/30 \) during the time interval \( t \in [0 \ 12.5]s \), and \( AM = \pi/15 \) during the time interval \( t \in [12.5 \ 25]s \), and finally with amplitude \( AM = 0 \) which represents a regulation case during the remaining time interval \( t \in [25 \ 30]s \). Fig 3 shows the tracking error converging rapidly to a value close to zero. Fig 4 represents the corresponding control input which peaks at \( t = 12.5 \) s (the first amplitude variation from \( AM = \pi/30 \) to \( AM = \pi/15 \)), and at \( t = 25 \) s (the second amplitude variation from \( AM = \pi/15 \) to \( AM = 0 \)). Fig 5 shows that the crisp estimated error provided by the fuzzy estimator is smooth, confirming the smoothing property of fuzzy logic systems. It also remains bounded and converges quite rapidly to a value close to zero. From these figures, we can see that the controlled system behaves well in all situations (tracking and regulation cases). Moreover it is practically able to eliminate disturbances introduced during all the elapsed time of test.

6. CONCLUSION

A SISO direct fuzzy adaptive control for a class of single input single output non linear systems has been introduced. A Takagi Sugeno inference system is used to approximate a feedback linearisation control law. The weight adaptation is derived from the minimisation of the control error and is based on a gradient descent algorithm. The unknown control
error appearing in the adaptation laws is replaced by an estimate provided by Mamdani fuzzy inference system. Global stability results based on Lyapunov theory are provided without the use of supervisor term in the control law. The algorithm is applied in simulation for the control of the inverted pendulum system for different situations. The results are encouraging. Moreover, this novel application of fuzzy logic systems confirms their smoothing capability as shown by the simulation results.

Fig. 5: The corresponding estimated control error $\hat{e}_u$

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