A 2D Systems Approach to Iterative Learning Control with Experimental Validation

Lukasz Hladowski * Zhonglun Cai ** Krzysztof Galkowski *
Eric Rogers ** Chris T Freeman ** Paul L Lewin **

* Institute of Control and Computation Engineering, University of Zielona Gora, Podgorna 50, 65-246 Zielona Gora, Poland
** School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK

Abstract: Iterative learning control is a technique especially developed for application to processes which are required to repeat the same operation, or task, over a finite duration and has many applications, such as a gantry robot undertaking a pick and place operation. The exact sequence is that once the task is completed, the robot is reset to the starting position and then the task is repeated. An iterative learning control algorithm exploits the fact that once an execution of the task is complete the control input and the output produced are available to update the control input for the next repetition with the aim of sequentially improving performance. This paper gives an overview of how control theory for 2D discrete linear systems and, in particular, repetitive processes can be used to design iterative learning control laws that deliver high quality performance in both simulation and experiments on a gantry robot executing a pick and place operation.

Keywords: iterative learning control, industrial robotics, 2D systems.

1. INTRODUCTION

Iterative Learning Control (ILC) is applicable to systems that operate in a repetitive manner where the task is to follow a predefined reference trajectory in a specified finite time interval, known as a pass or a trial in the literature, with high precision. The novel principle behind ILC is to suitably use data from previous trials, often in combination with appropriate current trial information, to select the current trial input to sequentially improve performance. In particular, the aim is to improve performance from trial-to-trial in the sense that the tracking error (the difference between the output on a trial and the specified reference trajectory) is sequentially reduced to either zero (ideal case) or some suitably small value.

The original work in this area is credited to Arimoto et al. (1984) and since then there have been substantial developments in both systems theoretic and applications terms. For an overview of the algorithm development see, for example, Ahn et al. (2007) which includes a categorization of algorithms developed pre 2004. Applications areas where ILC has been successfully applied include robotics, automated manufacturing plants and food processing. For more details, including some where there is clear potential for significant added benefit from fully developed ILC, one possible source is the survey article Briskov et al. (2006).

A fundamental systems theoretic problem in ILC is to determine the conditions under which trial-to-trial error convergence is guaranteed. This is an aspect of the general subject area which has seen much work for both linear and nonlinear plant models, including the target of monotonic error convergence in the trial-to-trial direction. In practice, zero tracking error is almost impossible to achieve due to random and non-repeating disturbances. Moreover, there is often a trade-off to be made between reduction of the trial-to-trial error and the performance achieved along the trials. For example, it is possible to converge trial-to-trial to a limit error which has unacceptable along the trial dynamics, see, for example, Owens et al. (2000). In essence, ILC is a 2D (information propagation in two directions) system and this paper gives an overview of recent work on the design of ILC laws in this setting with experimental verification on a gantry robot executing a pick and place operation that replicates many industrial applications such as food processing.

The symbols $M > 0$, respectively $M < 0$, are used in this paper to denote a symmetric positive definite, respectively negative, definite matrix.

2. CONTROL LAW DESIGN

The plants considered in this paper are assumed to be differential linear time-invariant systems described by the state-space triple $\{A, B, C\}$ which in an ILC setting is written as

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t), \quad 0 \leq t \leq \alpha$$
$$y_k(t) = Cx_k(t)$$

where on trial $k$, $x_k(t) \in \mathbb{R}^n$ is the state vector, $y_k(t) \in \mathbb{R}^m$ is the output vector, $u_k(t) \in \mathbb{R}^r$ is the vector of control inputs, and the trial length $\alpha < \infty$. If the signal to be
tracked is denoted by \( r(t) \) then \( e_k(t) = r(t) - y_k(t) \) is the error on trial \( k \). The most basic requirement then is to force the error to convergence in \( k \). This, however, cannot always be addressed independently of the dynamics along the trial. This, however, cannot always be addressed independently of the dynamics along the trial as the following analysis demonstrates.

Consider the case where on trial \( k+1 \) the control input is calculated using

\[
 u_{k+1}(t) := \sum_{j=1}^{M} a_j u_{k+1-j}(t) + \sum_{j=1}^{M} (K_j e_{k+1-j}(t) + (K_0 e_k + 1))
\]

(2)

In addition to the 'memory' \( M \), the design parameters in this control law are the static scalars \( a_j \), \( 1 \leq j \leq M \), the linear operator \( K_0 \) which describes the current pass error contribution, and the linear operator \( K_j \), \( 1 \leq j \leq M \), which describes the contribution from the error on pass \( k+1 - j \).

It is now routine to show that convergence of the error here holds if, and only if, all roots of

\[
 z^N - \alpha_1 z^{N-1} - \cdots - \alpha_{N-1} z - \alpha_N = 0 \tag{3}
\]

have modulus strictly less than unity. Also the error dynamics on trial \( k+1 \) here can be written in convolution form as

\[
 e_k(t) = r(t) - (Gu_{k+1})(t), 0 \leq t \leq \alpha
\]

Suppose also that (3) holds. Then the closed-loop error dynamics converge (in the norm topology of \( L_2[0, T] \)) to

\[
 e_\infty = (I + GK_0)^{-1} r
\]

(4)

where the so-called effective controller \( K_0 \) is given by

\[
 K_\text{eff} := \frac{K}{1 - \beta}
\]

and

\[
 \beta := \sum_{i=1}^{N} a_i, \quad K = \sum_{i=1}^{N} K_i
\]

The result here counter-intuitive in the sense that stability is largely independent of the plant and the controllers used. This is a direct result of the fact that the trial duration \( \alpha \) is finite and over such an interval a linear system can only produce a bounded output irrespective of its stability properties. Hence even if the error sequence generated is guaranteed to converge to a limit, this terminal error may be unstable and/or possibly worse than the first trial error, i.e. the use of ILC has produced no improvement in performance.

We have the following (see Owens et al. (2000) for the details).

1. Convergence is predicted to be ‘rapid’ if \( \lambda_e \) is small and will be geometric in form, converging approximately with \( \lambda_e^k \), where \( \lambda_e \in (\max|\mu_i|, 1) \) and \( \mu_i, 1 \leq i \leq N \), is a solution of (3).

2. The limit error is nonzero but is usefully described by a (1D linear systems) unity negative feedback system with effective controller \( K_\text{eff} \) defined above. If \( \max(|\mu_i|) \rightarrow 0+ \) then the limit error is essentially the first learning iterate, i.e. use of ILC has little benefit and will simply lead to the normal large errors encountered in simple feedback loops. There is hence pressure to let \( \max|\mu_i| \) be close to unity when \( K_\text{eff} \) is a high gain controller which will lead (roughly speaking) to small limit errors.

3. Zero limit error can only be achieved if \( \sum_{i=1}^{N} \alpha_i = 1 \). (This situation — again see Owens et al. (2000) — for the details — is reminiscent of classical control where the inclusion of an integrator (on the stability boundary) in the controller results in zero steady state (limit) error in response to constant reference signals.)

There is a conflict in the above conclusions which has implications on the systems and control structure from both the theoretical and practical points of view. In particular, consider for ease of presentation the case when \( K_i = 0, 1 \leq i \leq N \). Then small learning errors will require high effective gain yet \( GK_0 \) should be stable under such gains.

To guarantee an acceptable (i.e. stable (as the most basic requirement)) limit error and acceptable along the trial transients, a stronger form of stability must be used. Here we consider the use of so-called stability along the trial (or pass) from repetitive process theory. In effect, this requires convergence of the error sequence with a uniform bound on the along the trial dynamics. We also work in the discrete domain and so assume that the along the pass dynamics have been sampled at a uniform rate \( T_s \) seconds to produce a discrete-state space model of the form (where for notational simplicity the dependence on \( T_s \) is omitted from the variable descriptions)

\[
 x_k(p + 1) = Ax_k(p) + Bu_k(p), 0 \leq p \leq \alpha
\]

\[
 y_k(p) = Cx_k(p) \tag{5}
\]

Consider now the so-called discrete linear repetitive processes described by the following state-space model over \( p = 0, 1, \ldots, \alpha - 1, k \geq 1 \)

\[
 x_k(p+1) = Ax_k(p) + Bu_k(p) + B_0y_{k-1}(p)
\]

\[
 y_k(p) = Cx_k(p) + Du_k(p) + D_0y_{k-1}(p)
\]

where \( x_k(p) \in \mathbb{R}^n, u_k(p) \in \mathbb{R}^m, y_k(p) \in \mathbb{R}^m \) are the state, input and pass profile vectors respectively. Also rewrite the state equation of the process model in the form

\[
 x_k(p) = Ax_k(p-1) + Bu_k(p-1)
\]

and introduce

\[
 \eta_k(p+1) = x_k(p+1) - x_k(p)
\]

\[
 \Delta u_k(p+1) = u_k(p+1) - u_k(p) \tag{7}
\]

Then we have

\[
 \eta_k(p+1) = A \eta_k(p) + B \Delta u_k(p+1) \tag{8}
\]

Consider also a control law of the form

\[
 \Delta u_k(p+1) = K_1 \eta_k(p+1) + K_2 \varepsilon_k(p+1) \tag{9}
\]

and hence

\[
 \eta_k(p+1) = (A + BK_1) \eta_k(p) + BK_2 \varepsilon_k(p) \tag{10}
\]

Also \( \varepsilon_k(p+1) - \varepsilon_k(p) = y_k(p) - y_k(p+1) \) and we then obtain

\[
 e_k(p+1) - e_k(p) = CA(x_k(p+1) - x_k(p+1) - 1) + CBu_k(p+1) - u_k(p+1)
\]

(11)

Using (8) we now obtain

\[
 e_k(p+1) - e_k(p) = -CA \eta_k(p+1) - CB \Delta u_k(p+1) \tag{12}
\]

(13)
or, utilizing (10),

$$e_{k+1}(p) = -C(A + BK_1)\eta_{k+1}(p)$$

$$- C\eta_{k+1}(p) + (I - C BK_2)e_k(p)$$

(13)

Also introduce

$$\hat{A} = A + BK_1$$

$$\hat{B}_0 = BK_2$$

$$\hat{C} = -C(A + BK_1)$$

$$\hat{D}_0 = I - C BK_2$$

Then clearly (11) and (13) can be written as

$$\eta_{k+1}(p + 1) = \hat{A}\eta_{k+1}(p) + \hat{B}_0 e_k(p)$$

$$e_{k+1}(p) = \hat{C}\eta_{k+1}(p) + \hat{D}_0 e_k(p)$$

(15)

which is of the form (6) and hence the repetitive process stability theory can be applied to this ILC control scheme. In particular, stability along the trial is equivalent to uniform bounded input bounded output stability (defined in terms of the norm on the underlying function space), i.e. independent of the trial length, and hence we can (potentially) achieve trial-trial error convergence with acceptable along the trial dynamics.

The following result gives stability along the trial under control action together with formulas for control law design.

**Theorem 1.** The ILC scheme of (15) is stable along the trial if there exist compatibly dimensioned matrices $X_1 > 0$, $X_2 > 0$, $R_1$ and $R_2$ such that the following LMI is feasible

$$M = \begin{bmatrix}
-X_1 & 0 & -X_2 & 0 \\
0 & -X_2 & 0 & 0 \\
-CA_1 R_1 - CBR_1 & X_2 - CBR_2 & X_1 A^T + R_1^T B^T - X_1 A^T C^T - R_1^T B^T C^T & -X_1 \\
X_1 A^T & 0 & 0 & -X_2
\end{bmatrix} < 0$$

(16)

If (16) holds, the control law matrices $K_1$ and $K_2$ can be computed using

$$K_1 = R_1 X_1^{-1}$$

$$K_2 = R_2 X_2^{-1}$$

(17)

It is also possible to give a 2D discrete linear systems representation of the ILC scheme considered here, as first proposed in Kurek and Zaremba (1993). In particular, a necessary condition for stability along the trial is that all eigenvalues of the matrix $\hat{D}_0 = I - C BK_2$ have modulus strictly less than unity. This is precisely the condition obtained in Kurek and Zaremba (1993) using a Roesser state-space model. The work here progresses beyond this, and other work on ILC using 2D model descriptions, by use of a stronger form of stability, control law design and (uniquely for such ILC laws) experimental verification (see the next section).

One possible problem with the control law above is that it requires access to the current pass state vector and hence may need an observer for implementation. As an alternative, the same analysis base can be used to design control laws of the the form

$$u_k(p) = u_{k-1}(p) + K_1(y_k(p) - y_{k-1}(p))$$

$$+ K_2(y_k(p - 1) - y_{k-1}(p - 1))$$

$$+ K_3(y_{ref}(p + 1) - y_{k-1}(p + 1))$$

(18)

Here the last term is phase advance on the previous trial error, where in ILC such a term is well known in simple structure algorithms. Such an advance appears in the discrete-time implementation of the derivative ILC algorithm (Arimoto et al., 1985) where it was used to extend the applicability of the original ILC algorithm Arimoto et al. (1984). Since then, a variable advance has been considered. (Wang, 1999; Wang and Longman, 1996) and found to lead to accurate training in practice on a range of systems (Wallen et al., 2008). The second and third terms are proportional in nature acting on the error between the current and previous pass trials at $p$ and $p - 1$ respectively. Whilst use of current trial data has appeared in many approaches to manipulate the plant dynamics along the trial (Chen et al., 1996; Norrlöf and Gunnarsson, 1999) and has been found to increase initial tracking and disturbance rejection (Ratcliffe, 2005), the coupling of previous and current trial data is a novel addition to this class of updates. Implementation of this control law does not require a state observer (in repetitive process terms $y$ is the process output and hence available for measurement by definition) but does assume in this first work that the level of noise and other disturbances on the measurements is negligible. Note also that this control law design method covers both the multiple-input multiple-output (MIMO) and single-input single-output (SISO) cases.

### 3. EXPERIMENTS

Other work, e.g. Ratcliffe et al. (2006) has used a gantry robot facility to experimentally verify ILC designs. Figure 1 shows this experimental facility where the robot head performs a pick and place task and is similar to systems which can be found in many industrial applications. These include food canning, bottle filling or automotive assembly, all of which require accurate tracking control, each time the operation is performed, with a minimum level of error in order to maximize production rates. This is an obvious general area for application of ILC.

Each axis of the gantry robot has been modelled based on frequency response tests where, since the axes are orthogonal, it is assumed that there is minimal interaction between them. Here we first consider the X-axis (the one parallel to the conveyor in Figure 1) and frequency response tests (via the Bode gain and phase plots in Figure 2) result in a 7th order continuous-time transfer-function

Fig. 1. The multi-axes gantry robot.
Consider the case when the uncertainty is modeled as additive uncertainty in the process state-space matrices of the form

\[
A = \tilde{A} + \Delta A \\
B = \tilde{B} + \Delta B
\]  \hspace{1cm} (19)

where \( \tilde{A} \) and \( \tilde{B} \) represent the nominal versions of \( A \) and \( B \) respectively and \( \Delta A \) and \( \Delta B \) are uncertainties that by assumption satisfy

\[
\Delta A = H_1 F_1 E_1 \\
\Delta B = H_2 F_2 E_2
\]

where \( F_1 = F_1^T \), \( F_1^T F_1 \preceq I \), \( F_2 = F_2^T \), \( F_2^T F_2 \preceq I \) and \( F_1 \in \mathbb{R}^{n \times n} \), \( F_2 \in \mathbb{R}^{n \times n} \), \( H_1 \), \( E_1 \), \( H_2 \), \( E_2 \) are matrices of appropriate dimensions. Here \( H_1 \), \( H_2 \), \( E_1 \) and \( E_2 \) are assumed constant but \( F_1 \) and \( F_2 \) can vary both from trial-to-trial and from point-to-point along the trial, but under the requirement that they are norm bounded. Here we aim to control the ILC dynamics by a linear control law which only uses plant output information.

It is easy to see that in this case

\[
e_{k+1}(p) = e_k(p) = CA(-x_{k+1}(p-1) + x_k(p-1)) \\
+ CB(-u_{k+1}(p-1) + u_k(p-1))
\]  \hspace{1cm} (20)

Also introduce

\[
\eta_{k+1}(p+1) = x_{k+1}(p) - x_k(p) \\
\Delta u_{k+1}(p) = u_{k+1}(p) - u_k(p)
\]  \hspace{1cm} (21)

and consider a control law of the form

\[
\Delta u_{k+1}(p) = K_1 \mu_{k+1}(p+1) + K_2 \mu_{k+1}(p) + K_3 e_k(p+1)
\]  \hspace{1cm} (22)

where

\[
\mu_k(p) = y_k(p-1) - y_{k-1}(p-1) = C \eta_k(p)
\]  \hspace{1cm} (23)

Now introduce

\[
\eta_{k+1}(p+1) = \hat{\Lambda} \eta_{k+1}(p) + \hat{B}_0 e_k(p) \\
e_{k+1}(p) = \hat{C} \eta_{k+1}(p) + \hat{D}_0 e_k(p)
\]  \hspace{1cm} (24)

where

\[
\hat{\Lambda} = \begin{bmatrix} \hat{A} + \Delta A & (\hat{B} + \Delta B)K_1C \\ I & 0 \end{bmatrix} \\
\hat{B}_0 = \begin{bmatrix} (\hat{B} + \Delta B)K_3 \\ 0 \end{bmatrix} \\
\hat{C} = \begin{bmatrix} -CA - CBK_1C \end{bmatrix} \\
\hat{D}_0 = \begin{bmatrix} (I - CBK_3) \end{bmatrix}
\]  \hspace{1cm} (25)

which is of the form (15) and hence the repetitive process stability theory can be applied to this ILC control scheme.

We also consider uncertainty defined by the following matrices, which move at least one eigenvalue of the matrix \( A \) outside the unit circle and hence a necessary condition for stability along the trial is violated.
Trial Number
Mean Squared Error (mm$^2$)

0.0018961 0.0041933 0.0001093 0.0016410
0.0037826 0.0047463 0.004926 0.0045098
0.0042366 0.0008206 0.0047892 0.0047460

$H_1 =
\begin{bmatrix}
0.0016476 & 0.0009006 & 0.001099 & 0.004630 \\
0.0010935 & 0.0010321 & 0.002735 & 0.0065948 \\
0.0049923 & 0.0049409 & 0.001418 & 0.0000427 \\
0.0003872 & 0.0005230 & 0.004644 & 0.0009791 \\
\end{bmatrix}$

0.0046033 0.0030797 0.0010673
0.0005337 0.0002871 0.0017437
0.0026180 0.0033403 0.0047795
0.0018489 0.0049765 0.0020856
0.0000740 0.0019517 0.0013230
0.0034262 0.0047406 0.0023793

$H_2 =
\begin{bmatrix}
0.0049660 & 0.003496 & 0.0021375 & 0.0008747 \\
0.0004483 & 0.0021330 & 0.0017147 & 0.0010256 \\
0.0047179 & 0.0044222 & 0.0017134 & 0.0035423 \\
0.0046799 & 0.0043186 & 0.0016616 & 0.0034730 \\
0.0034889 & 0.0004317 & 0.0004343 & 0.0014831 \\
0.0036245 & 0.0046470 & 0.0009547 & 0.0001948 \\
0.0003798 & 0.0010466 & 0.0005933 & 0.0027058 \\
0.0035732 & 0.0039283 & 0.0032364 & 0.0046809 \\
\end{bmatrix}$

$E_1 = \begin{bmatrix}
0.0614517 & 0.0236343 & 0.0917077 & 0.0191017 \\
0.0057852 & 0.0622740 & 0.0316540 & 0.0389425 \\
0.0006452 & 0.0311047 & 0.0415206 & 0.0991693 \\
0.0296296 & 0.0741962 & 0.0546119 & 0.0889646 \\
0.0407110 & 0.0998781 & 0.0277466 & 0.0877306 \\
0.0271111 & 0.0002268 & 0.0938016 & 0.0979093 \\
0.0153410 & 0.0621013 & 0.0816633 & 0.0059753 \\
\end{bmatrix}$

$E_2 = \begin{bmatrix}
0.0342115 & 0.0490774 & 0.0272181 \\
0.0794277 & 0.0951360 & 0.0425609 \\
0.0575884 & 0.0923998 & 0.0300095 \\
0.0304417 & 0.0193673 & 0.0358908 \\
0.0326482 & 0.0248840 & 0.0417899 \\
0.0936791 & 0.0353672 & 0.0849825 \\
0.0825843 & 0.0792671 & 0.0786548 \\
\end{bmatrix}$

and both $F_1$ and $F_2$ are taken as a $7 \times 7$ diagonal matrix with each non-zero entry set at 0.99. A necessary condition for stability along the trial is that the state matrix in the plant model must have spectral radius less than unity. For this perturbation this spectral radius is equal to $1.0017$ and applying the design algorithm in this case gives

$K_1 = -121.53473, \quad K_2 = -13.884311, \quad K_3 = 57.45916$

(26)

Figure 4 shows the progression of the input, output and error as the trials are completed and Fig 5 the mean squared error against trial number. These show that the objectives of trial-to-trial error convergence and along the trial performance can both be well controlled and further refinement to, for example, place stronger emphasis on the mean squared error reduction is possible.

![Fig. 5. Experimentally measured mean squared error plotted against trial number.](image)

4. CONCLUSIONS

This paper has given an overview of recent results on the use of 2D control systems theory in the design of linear iterative learning control. The major conclusion is that this approach, which has been known since the early 1990s, can be extended to allow control law design for both trial-to-trial error convergence and also performance along the trials. The resulting conditions can also be computed using LMIs and the control laws have a well defined physical structure. A significant feature here is the very good agreement between the results predicted by the theory and those measured experimentally from the gantry robot system.

It is also possible to extend this analysis to deal with uncertainty in the plant model where again there is excellent agreement between predicted and measured results. These results are the first ever on robust iLC in a 2D systems setting to have any form of experimental verification and argue strongly for an expanded programme of research in this area.

REFERENCES


Fig. 4. Experimentally measured input (left), output (center) and error (right) progression with trial number.


