Control and reachability of non-homogeneous wave equation related objects

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Model equations

Considered are two ways of a homogeneous wave equation generalization

$$\ddot{y}(x,t) - \kappa y''(x,t) = 0 \quad t \geq 0,\ 0 \leq x \leq 1$$

Here, we adopt the notation $\dot{g} = g_t$, $g' = g_x$.

non-homogeneous string equation

$$\ddot{v}(x,t) - a(x)v''(x,t) = 0$$

hanging chain equation

$$\ddot{z}(x,t) - (a(x)z'(x,t))' = 0$$

a is a real function bounded by 2 positive constants.

Dirichlet conditions

$$y(0,t) = y(1,t) = 0$$

Neumann conditions

$$y'(0,t) = y'(1,t) = 0$$

mixed Dirichlet-Neumann conditions

$$y(0,t) = y'(1,t) = 0$$

mixed Neumann-Dirichlet conditions

$$y'(0,t) = y(1,t) = 0$$

for $t \geq 0$.

Operators

string

We consider the space H, whose underlying set is $L^2(0,1)$ and with the inner product defined by formula

$$< y_1, y_2 >= \int_0^1 \frac{1}{a(x)} y_1(x) y_2(x) dx$$

hanging chain

We consider the space $H = L^2(0,1)$ with the standard inner product.

Dirichlet

$$D(A) = \{y \in H : y(0) = y(1) = 0\}$$

Neumann

$$D'(A) = \{y \in H : y'(0) = y'(1) = 0\}$$

Remark

$D(A)$ and $D'(A)$ are dense in $H^2(0,1)$: the space of all functions in H that are twice differentiable and the second derivative is again in H.

string

$$A : D(A) \to H \quad \text{or} \quad A : D'(A) \to H \quad Ay = -ay''$$

chain

$$B : D(A) \to H \quad \text{or} \quad B : D'(A) \to H \quad By = -(ay')'$$

Properties of the operators

The operators A and B are

- positive,
- symmetric,
- invertible and their range is H, (Dirichlet only)
- self-adjoint and have compact resolvent.

Spectral properties

string

$$-a(x)v''(x) = \lambda v(x)$$

chain

$$-(a(x)z'(x))' = \lambda z(x)$$

Theorem

The solutions of the spectral equations with boundary conditions $v(0) = z(0) = 0$ are

$$v(x) = C \exp\left(\frac{1}{4}a(x)\right) \sin\sqrt{\lambda} \int_0^x \frac{ds}{\sqrt{a(s)}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

$$z(x) = C \exp\left(-\frac{1}{4}a(x)\right) \sin\sqrt{\lambda} \int_0^x \frac{ds}{\sqrt{a(s)}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

Theorem

The solutions of the spectral equations with boundary conditions $v'(0) = z'(0) = 0$ are

$$v(x) = C\exp\left(\frac{1}{4}a(x)\right)\cos\sqrt{\lambda}\int_0^x \frac{ds}{\sqrt{a(s)}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

$$z(x) = C\exp\left(-\frac{1}{4}a(x)\right)\cos\sqrt{\lambda}\int_0^x \frac{ds}{\sqrt{a(s)}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

Theorem

The eigenvalues of Dirichlet operators A and B are

$$\lambda_n = \left(\int_0^1 \frac{ds}{\sqrt{a(s)}} \right)^{-2} (n\pi + O(1/n))^2$$

where n is a positive integer and beginning from sufficiently large n_0, the eigenvalues are simple.

Theorem

The eigenvalues of Neumann operators A and B are

$$\lambda_n = \left(\int_0^1 \frac{ds}{\sqrt{a(s)}} \right)^{-2} (n\pi + O(1/n))^2$$

where n is a positive integer and beginning from sufficiently large n_0, the eigenvalues are simple. We also have simple eigenvalue $\lambda_0 = 0$.

Theorem

The almost normalized eigenvectors of the operator A are

$$v_n(x) = \frac{\sqrt{2}\exp\left(\frac{1}{4}a(x)\right)}{\left(\int_0^1 \frac{\exp\left(\frac{1}{2}a(t)\right)}{a(t)}dt\right)^{1/2}} \sin J(\lambda_n, x) + O\left(\frac{1}{n}\right)$$

and the almost normalized eigenvectors of the operator B are

$$z_n(x) = \frac{\sqrt{2}\exp\left(-\frac{1}{4}a(x)\right)}{\left(\int_0^1 \exp\left(-\frac{1}{2}a(t)\right)dt\right)^{1/2}} \sin J(\lambda_n, x) + O\left(\frac{1}{n}\right)$$

where n is a positive integer and

$$J(\lambda_n, x) = \sqrt{\lambda_n}\int_0^x \frac{ds}{\sqrt{a(s)}}.$$

Theorem

The almost normalized eigenvectors of the operator A are

$$v_0(x) = \left(\int_0^1 \frac{ds}{a(s)} \right)^{-1/2}$$

$$v_n(x) = \frac{\sqrt{2} \exp\left(\frac{1}{4} a(x)\right)}{\left(\int_0^1 \frac{\exp\left(\frac{1}{2} a(t)\right)}{a(t)} dt \right)^{1/2}} \cos J(\lambda_n, x) + O\left(\frac{1}{n}\right)$$

where n is a positive integer and

$$J(\lambda_n, x) = \sqrt{\lambda_n} \int_0^x \frac{ds}{\sqrt{a(s)}}.$$

Theorem

The almost normalized eigenvectors of the operator B are

$$z_0(x) = 1$$
$$z_n(x) = \frac{\sqrt{2}\exp\left(-\frac{1}{4}a(x)\right)}{\left(\int_0^1 \exp\left(-\frac{1}{2}a(t)\right)dt\right)^{1/2}} \cos J(\lambda_n, x) + O\left(\frac{1}{n}\right)$$

where n is a positive integer and

$$J(\lambda_n, x) = \sqrt{\lambda_n}\int_0^x \frac{ds}{\sqrt{a(s)}}.$$

Controllability problem

String

Equation: $\ddot{v}(x,t) - a(x)v''(x,t) = 0$
Boundary conditions (Dirichlet): $v(0,t) = 0$, $v(1,t) = u(t)$
Boundary conditions (Neumann): $v'(0,t) = 0$, $v'(1,t) = u(t)$
Initial conditions: $v(x,0) = 0$, $\dot{v}(x,0) = 0$
End conditions: $v(x,T) = y_T(x)$, $\dot{v}(x,T) = \dot{y}_T(x)$

Chain

Equation: $\ddot{z}(x,t) - (a(x)z'(x,t))' = 0$
Boundary conditions (Dirichlet): $z(0,t) = 0$, $z(1,t) = u(t)$
Boundary conditions (Neumann): $z'(0,t) = 0$, $z'(1,t) = u(t)$
Initial conditions: $z(x,0) = 0$, $\dot{z}(x,0) = 0$
End conditions: $z(x,T) = y_T(x)$, $\dot{z}(x,T) = \dot{y}_T(x)$

Controllability problem

Describe the function u that allows to control the object from the initial state $(0, 0)$ to the end state $(y_T, \dot{y}_T)$ in given time T.

Substitution

$\varphi(x,t) = \varphi_y(x,t) + f(x)u(t)$
$\varphi(x,t)$ is either z or v

string

Dirichlet: $f(x) = x$, Neumann: $f(x) = \frac{1}{2}x^2$

chain

Dirichlet: $f(x) = a(1)\int_0^x \frac{ds}{a(s)}$, Neumann: $f(x) = a(1)\int_0^x \frac{sds}{a(s)}$

String

Equation (Dirichlet): $\ddot{v}_y(x,t) - a(x)v_y''(x,t) = -x\ddot{u}(t)$
Equation (Neumann):
$\ddot{v}_y(x,t) - a(x)v_y''(x,t) = -\frac{1}{2}x^2\ddot{u}(t) + a(x)u(t)$
Boundary conditions (Dirichlet): $v_y(0,t) = 0$, $v_y(1,t) = 0$
Boundary conditions (Neumann): $v_y'(0,t) = 0$, $v_y'(1,t) = 0$
Initial conditions: $v_y(x,0) = -f(x)u(0)$, $\dot{v}_y(x,0) = -f(x)\dot{u}(0)$
End conditions: $v_y(x,T) = y_T(x) - f(x)u(T)$,
$\dot{v}_y(x,T) = \dot{y}_T(x) - f(x)\dot{u}(T)$

Chain

Equation (Dirichlet): $\ddot{z}_y(x,t) - (a(x)z_y'(x,t))' = -x\ddot{u}(t)$
Equation (Neumann):
$\ddot{z}_y(x,t) - (a(x)z_y'(x,t))' = -\frac{1}{2}x^2\ddot{u}(t) + a(1)u(t)$
Boundary conditions (Dirichlet): $z_y(0,t) = 0$, $z_y(1,t) = 0$
Boundary conditions (Neumann): $z_y'(0,t) = 0$, $z_y'(1,t) = 0$
Initial conditions: $z_y(x,0) = -f(x)u(0)$, $\dot{z}_y(x,0) = -f(x)\dot{u}(0)$
End conditions: $z_y(x,T) = y_T(x) - f(x)u(T)$,
$\dot{z}_y(x,T) = \dot{y}_T(x) - f(x)\dot{u}(T)$

String

The unique weak solution of the string equation with initial conditions $v_y(x,0) = -f(x)u(0)$, $\dot{v}_y(x,0) = -f(x)\dot{u}(0)$ is

$$v_y(x,t) = \sum_{n=1}^{\infty} \left((-1)^{n+1} \hat{c}_n \int_0^t u(s) \sin\left(\sqrt{\lambda_n}(t-s)\right) ds \right) v_n(x)$$

where v_n is an eigenvector of the operator A with the corresponding eigenvalue λ_n and $\hat{c}_n$ is the sequence converging to $\left(\frac{a(1)}{2}\int_0^1 \frac{e^{a(t)/2}dt}{a(t)}\right)^{-1/2} e^{a(1)/4}$.

Chain

The unique weak solution of the hanging chain equation with initial conditions $z_y(x,0) = -f(x)u(0)$, $\dot{z}_y(x,0) = -f(x)\dot{u}(0)$ is

$$z_y(x,t) = \sum_{n=1}^{\infty} \left((-1)^{n+1} \hat{c}_n \int_0^t u(s) \sin\left(\sqrt{\lambda_n}(t-s)\right) ds \right) z_n(x)$$

where z_n is an eigenvector of the operator B with the corresponding eigenvalue λ_n and $\hat{c}_n$ is the sequence converging to $\left(\frac{1}{2a(1)} \int_0^1 e^{-a(t)/2} dt\right)^{-1/2} e^{-a(1)/4}$.

We insert the end conditions into the weak solution, make inner product with eigenvectors on both sides and use orthonormality of eigenvectors. We obtain for $n > 0$ what follows.

Trigonometric moment problem

$$\int_0^T u(t)\cos\left(\sqrt{\lambda_n}t\right)dt = \dot{c}_n$$
$$\int_0^T u(t)\sin\left(\sqrt{\lambda_n}t\right)dt = c_n$$

with

$$\dot{c}_n = \frac{(-1)^{n+1}}{\hat{c}_n}\left(\langle y_T,\varphi_n\rangle \sin\left(\sqrt{\lambda_n}T\right) + \frac{\langle \dot{y}_T,\varphi_n\rangle}{\sqrt{\lambda_n}}\cos\left(\sqrt{\lambda_n}T\right)\right)$$
$$c_n = \frac{(-1)^{n+1}}{\hat{c}_n}\left(-\langle y_T,\varphi_n\rangle \cos\left(\sqrt{\lambda_n}T\right) + \frac{\langle \dot{y}_T,\varphi_n\rangle}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}T\right)\right)$$

where φ_n's are either v_n's or z_n's.

Exponential moment problem

$$\int_0^T u(t) \exp t\sqrt{\lambda_n} dt = b_n$$

with $b_n = c_n + \dot{c}_n$ for $n > 0$ and $b_n = c_n - \dot{c}_n$ for $n < 0$.

Controllability problem revisited

Describe the function u that satisfies the above moment problem for $T > 0$.

String

The unique weak solution of the string equation with initial conditions $v_y(x,0) = -f(x)u(0)$, $\dot{v}_y(x,0) = -f(x)\dot{u}(0)$ is

$$v_y(x,t) = \left(\int_0^1 \frac{ds}{a(s)}\right)^{-1} \int_0^t (t-s)u(s)\,ds + \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{\lambda_n}} \left(\hat{c}_n' \int_0^t u(s) \sin\left(\sqrt{\lambda_n}(t-s)\right) ds\right) v_n(x),$$

where v_n is an eigenvector of the operator A with the corresponding eigenvalue λ_n and $\hat{c}_n'$ is the sequence converging to $\left(\frac{1}{2}\int_0^1 \frac{e^{a(t)/2}dt}{a(t)}\right)^{-1/2} e^{a(1)/4}$.

Chain

The unique weak solution of the hanging chain equation with initial conditions $z_y(x,0) = -f(x)u(0)$, $\dot{z}_y(x,0) = -f(x)\dot{u}(0)$ is

$$z_y(x,t) = a(1)\int_0^t (t-s)u(s)\,ds + \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{\lambda_n}} \left(\hat{c}'_n \int_0^t u(s) \sin\left(\sqrt{\lambda_n}(t-s)\right) ds \right) z_n(x),$$

where z_n is an eigenvector of the operator B with the corresponding eigenvalue λ_n and $\hat{c}'_n$ is the sequence converging to $a(1)\left(\frac{1}{2}\int_0^1 e^{-a(t)/2}dt\right)^{-1/2} e^{-a(1)/4}$.

Trigonometric moment problem

$$\int_0^T u(t)dt = \dot{c}_0',$$

$$\int_0^T u(t)\cos\left(\sqrt{\lambda_n}t\right)dt = \dot{c}_n' \qquad \text{for } n > 0,$$

$$\int_0^T tu(t)dt = c_0',$$

$$\int_0^T u(t)\sin\left(\sqrt{\lambda_n}t\right)dt = c_n' \qquad \text{for } n > 0,$$

with

$$\dot{c}_0' = \frac{1}{\hat{c}_0'} \langle \dot{y}_T, \varphi_0 \rangle ,$$

$$\dot{c}_n' = \frac{(-1)^n}{\hat{c}_n'} \left(\langle y_T, \varphi_n \rangle \sqrt{\lambda_n} \sin\left(\sqrt{\lambda_n} T\right) + \langle \dot{y}_T, \varphi_n \rangle \cos\left(\sqrt{\lambda_n} T\right)\right) ,$$

$$c_0' = \frac{1}{\hat{c}_0'} \left(- \langle y_T, \varphi_0 \rangle + T \langle \dot{y}_T, \varphi_0 \rangle \right) ,$$

$$c_n' = \frac{(-1)^n}{\hat{c}_n} \left(-\langle y_T, \varphi_n \rangle \sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n} T\right) + \langle \dot{y}_T, \varphi_n \rangle \sin\left(\sqrt{\lambda_n} T\right)\right)$$

where φ_n's are either v_n's or z_n's.

Exponential moment problem

$$\int_0^T u(t)dt = \dot{c}_0',$$

$$\int_0^T tu(t)dt = c_0',$$

$$\int_0^T u(t)\exp t\sqrt{\lambda_n}dt = b_n$$

with $b_n = c_n + \dot{c}_n$ for $n > 0$ and $b_n = c_n - \dot{c}_n$ for $n < 0$.

Controllability problem revisited

Describe the function u that satisfies the above moment problem for $T > 0$.

Solution, Dirichlet

Assume $T \geq 2\int_0^1 \frac{dx}{\sqrt{a(x)}}$. There exists a function u that satisfies the moment problem equations, if and only if the series

$$\sum_{n=-\infty}^{\infty} b_n^2 \quad \text{converges.}$$

Solution, Neumann

Assume $T > 2\int_0^1 \frac{dx}{\sqrt{a(x)}}$. There exists a function u that satisfies the moment problem equations, if and only if the series

$$\sum_{n=-\infty}^{\infty} b_n^2 \quad \text{converges.}$$

Statement of the problem

Describe the space of all final states that can be reached from the state $(0, 0)$ with the use of suitable control function u in given time T.

Solution, Dirichlet

Assume $T \geq 2\int_0^1 \frac{dx}{\sqrt{a(x)}}$. The space of all reachable states is $L^2(0,1) \times W^{-1}(0,1)$, where $W^{-1}(0,1)$ is the space of all functions from $L^2(0,1)$, whose primitives are again in $L^2(0,1)$.

Solution, Neumann

Assume $T > 2\int_0^1 \frac{dx}{\sqrt{a(x)}}$. The space of all reachable states is $W^1(0,1) \times L^2(0,1)$, where $W^1(0,1)$ is the space of all functions from $L^2(0,1)$, whose derivatives are again in $L^2(0,1)$.

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Thank you for attention.